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$SL(2)$ and z-measures

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1 Introduction

This paper is about the z-measures which are a remarkable two-parametric family of measures on partitions introduced in [10] in the context of harmonic analysis on the infinite symmetric group. In a series of papers, A. Borodin and G. Olshanski obtained several fundamental results on these z-measures, see their survey [5] which appears in this volume and also [3]. The culmination of this development is an exact determinantal formula for the correlation functions of the z-measures in terms of the hypergeometric kernel [4]. We mention [2] as one of the applications of this formula. The main result of this paper is a representation-theoretic derivation of the formula of Borodin and Olshanski.

1.1

In the early days of z-measures, it was already noticed that z-measures have some mysterious connection to representation theory of $SL(2)$. For example, the z-measure is actually positive if its two parameters z and z' are either complex conjugate $z' = \bar{z}$ or $z, z' \in (n, n+1)$ for some $n \in \mathbb{Z}$. In these cases $z - z'$ is either imaginary or lies in $(-1, 1)$, which was certainly reminiscent of the principal and complementary series of representations of $SL(2)$.

Later, S. Kerov constructed an $SL(2)$-action on partitions for which the z-measures are certain matrix elements [9]. Finally, Borodin and Olshanski computed the correlation functions of the z-measures is in terms of the Gauss hypergeometric function which is well known to arise as matrix elements

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of representations of $SL(2)$. The aim of this paper is to put these pieces together.

1.2

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The constructions of this paper were subsequently generalized beyond $SL(2)$ and z-measures in [12].

2 The z-measures, Kerov's operators, and correlation functions

2.1 Definition of the z-measures

Let $z, z' \in \mathbb{C}$ be two parameters and consider the following measure on the set of all partitions λ of n

$$\mathcal{M}_n(\lambda) = \frac{n!}{(zz')_n} \prod_{\square \in \lambda} \frac{(z + c(\square))(z' + c(\square))}{h(\square)^2}\,, \tag{2.1}$$

where

$$(x)_n = x(x+1)\dots(x+n-1)\,,$$

the product is over all squares $\square$ in the diagram of λ, $h(\square)$ is the length of the corresponding hook, and $c(\square)$ stands for the content of the square $\square$. Recall that, by definition, the content of $\square$ is

$$c(\square) = \text{column}(\square) - \text{row}(\square)\,,$$

where column$(\square)$ denotes the column number of the square $\square$. The reader is referred to [11] for general facts about partitions.

It is not immediately obvious from the definition (2.1) that

$$\sum_{|\lambda|=n} \mathcal{M}_n(\lambda) = 1\,. \tag{2.2}$$

One possible proof of (2.2) uses the following operators on partitions introduced by S. Kerov.

2.2 Kerov's operators

Consider the vector space with an orthonormal basis $\{\delta_\lambda\}$ indexed by all partitions of λ of any size. Introduce the following operators

$$\begin{aligned} U\,\delta_\lambda &= \sum_{\mu=\lambda+\square} (z+c(\square))\,\delta_\mu \\ L\,\delta_\lambda &= \qquad (zz'+2|\lambda|)\,\delta_\lambda \\ D\,\delta_\lambda &= \sum_{\mu=\lambda-\square} (z'+c(\square))\,\delta_\mu\,, \end{aligned}$$

where $\mu=\lambda+\square$ means that μ is obtained from λ by adding a square $\square$ and $c(\square)$ is the content of this square. The letters U and D here stand for "up" and "down".

These operators satisfy the commutation relations

$$[D,U]=L\,, \quad [L,U]=2U\,, \quad [L,D]=-2D\,, \tag{2.3}$$

same as for the following basis of $\mathfrak{sl}(2)$

$$U=\begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad L=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \quad D=\begin{pmatrix}0&0\\-1&0\end{pmatrix}.$$

In particular, it is clear that if $|\lambda|=n$ then

$$(U^n\delta_\emptyset,\delta_\lambda)=\dim\lambda\prod_{\square\in\lambda}(z+c(\square))$$

where

$$\dim\lambda=n!\prod_{\square\in\lambda}h(\square)^{-1}$$

is the number of standard tableaux on λ. It follows that

$$\mathcal{M}_n(\lambda)=\frac{1}{n!\,(zz')_n}\,(U^n\delta_\emptyset,\delta_\lambda)\,(L^n\delta_\lambda,\delta_\emptyset)\,.$$

Using this presentation and the commutation relations (2.3) one proves (2.2) by induction on n.

2.3 The measure $\mathcal{M}$ and its normalization

In a slightly different language, with induction on n replaced by the use of generating functions, this computation goes as follows.

The sequence of the measures $\mathcal{M}_n$ can be conveniently assembled as in [4] into one measure $\mathcal{M}$ on the set of all partitions of all numbers as follows

$$\mathcal{M} = (1-\xi)^{zz'} \sum_{n=0}^{\infty} \xi^n \, \frac{(zz')_n}{n!} \, \mathcal{M}_n \,, \quad \xi \in [0,1) \,,$$

where ξ is a new parameter. In other words, $\mathcal{M}$ is the mixture of the measures $\mathcal{M}_n$ by means of a negative binomial distribution on n with parameter ξ.

It is clear that (2.2) is now equivalent to $\mathcal{M}$ being a probability measure. It is also clear that

$$\mathcal{M}(\lambda) = (1-\xi)^{zz'} (e^{\sqrt{\xi}\, U} \, \delta_\emptyset, \delta_\lambda) \, (e^{\sqrt{\xi}\, D} \, \delta_\lambda, \delta_\emptyset) \tag{2.4}$$

Therefore

$$\sum_\lambda \mathcal{M}(\lambda) = (1-\xi)^{zz'} (e^{\sqrt{\xi}\, D} \, e^{\sqrt{\xi}\, U} \, \delta_\emptyset, \delta_\emptyset) \tag{2.5}$$

It follows from the definitions that

$$D \, \delta_\emptyset = 0 \,, \quad L \, \delta_\emptyset = zz' \, \delta_\emptyset \,, \quad U^* \, \delta_\emptyset = 0 \,, \tag{2.6}$$

where U^* is the operator adjoint to U. Therefore, in order to evaluate (2.5), it suffices to commute $e^{\sqrt{\xi}\, L}$ through $e^{\sqrt{\xi}\, U}$.

The following computation in the group $SL(2)$

$$\begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\alpha}{1-\alpha\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\alpha\beta} & 0 \\ 0 & 1-\alpha\beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\beta}{1-\alpha\beta} & 1 \end{pmatrix}$$

implies that

$$\exp(\beta\, D) \, \exp(\alpha\, U) = \exp\left(\frac{\alpha}{1-\alpha\beta} \, U \right) (1-\alpha\beta)^{-L} \exp\left(\frac{\beta}{1-\alpha\beta} \, D \right) , \tag{2.7}$$

provided $|\alpha\beta| < 1$. Therefore,

$$\begin{aligned} \sum_\lambda \mathcal{M}(\lambda) &= (1-\xi)^{zz'} \left(\exp\left(\frac{\sqrt{\xi}}{1-\xi} \, U \right) (1-\xi)^{-L} \, \exp\left(\frac{\sqrt{\xi}}{1-\xi} \, D \right) \delta_\emptyset, \delta_\emptyset \right) \\ &= (1-\xi)^{zz'} \left((1-\xi)^{-L} \, \delta_\emptyset, \delta_\emptyset \right) = 1 \,, \end{aligned}$$

as was to be shown.

2.4 Correlation functions

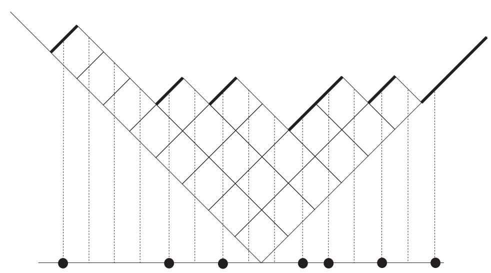
Introduce the following coordinates on the set of partitions. To a partition λ we associate a subset

$$\mathfrak{S}(\lambda) = \{\lambda_i - i + 1/2\} \subset \mathbb{Z} + \tfrac{1}{2}\,.$$

For example,

$$\mathfrak{S}(\emptyset) = \left\{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\right\}$$

This set $\mathfrak{S}(\lambda)$ has the following geometric interpretation. Take the diagram of λ and rotate it 135° as in the following picture:



The positive direction of the axis points to the left in the above figure. The boundary of λ forms a zigzag path and the elements of $\mathfrak{S}(\lambda)$, which are marked by $\bullet$, correspond to moments when this zigzag goes up.

Subsets $S \subset \mathbb{Z} + \frac{1}{2}$ of the form $S = \mathfrak{S}(\lambda)$ can be characterized by

$$|S_+| = |S_-| < \infty$$

where

$$S_+ = S \setminus \left(\mathbb{Z}_{\le 0} - \tfrac{1}{2}\right)\,, \quad S_- = \left(\mathbb{Z}_{\le 0} - \tfrac{1}{2}\right) \setminus S\,.$$

The number $|\mathfrak{S}_+(\lambda)| = |\mathfrak{S}_-(\lambda)|$ is the number of squares in the diagonal of the diagram of λ and the finite set $\mathfrak{S}_+(\lambda) \cup \mathfrak{S}_-(\lambda) \subset \mathbb{Z} + \frac{1}{2}$ is known as the modified Frobenius coordinates of λ.

Given a finite subset $X \in \mathbb{Z} + \frac{1}{2}$, define the *correlation function* by

$$\rho(X) = \mathcal{M}\big(\{\lambda, X \subset \mathfrak{S}(\lambda)\}\big)\,.$$

In [4], A. Borodin and G. Olshanski proved that

$$\rho(X) = \det\Big[K(x_i, x_j)\Big]_{x_i, x_j \in X}$$

where K the *hypergeometric kernel* introduced in [4]. This kernel involves the Gauss hypergeometric function and the explicit formula for K will be reproduced below.

It is our goal in the present paper to give a representation-theoretic derivation of the formula for correlation functions and, in particular, show how the kernel K arises from matrix elements of irreducible $SL(2)$-modules.

3 SL(2) and correlation functions

3.1 Matrix elements of $\mathfrak{sl}(2)$-modules and Gauss hypergeometric function

The fact that the hypergeometric function arises as matrix coefficients of $SL(2)$ modules is well known. A standard way to see this is to use a functional realization of these modules; the computation of matrix elements leads then to an integral representation of the hypergeometric function, see for example how matrix elements of $SL(2)$-modules are treated in [15]. An alternative approach is to use explicit formulas for the action of the Lie algebra $\mathfrak{sl}(2)$ and it goes as follows.

Consider the $\mathfrak{sl}(2)$-module V with the basis v_k indexed by all half-integers $k \in \mathbb{Z} + \frac{1}{2}$ and the following action of $\mathfrak{sl}(2)$

$$U\,v_k = (z + k + \tfrac{1}{2})\,v_{k+1}\,, \tag{3.1}$$

$$L\,v_k = (2k + z + z')\,v_k\,, \tag{3.2}$$

$$D\,v_k = (z' + k - \tfrac{1}{2})\,v_{k-1}\,. \tag{3.3}$$

It is clear that

$$e^{\alpha U}\,v_k = \sum_{s=0}^{\infty} \frac{\alpha^s}{s!}\,(z + k + \tfrac{1}{2})_s\,v_{k+s}\,.$$

Introduce the following notation

$$(a)_{\downarrow s} = a(a-1)(a-2)\cdots(a-s+1)\,.$$

With this notation we have

$$e^{\beta D}\,v_k = \sum_{s=0}^{\infty} \frac{\beta^s}{s!}\,(z' + k - \tfrac{1}{2})_{\downarrow s}\,v_{k-s}\,.$$

Denote by $[i \to j]_{\alpha,\beta,z,z'}$ the coefficient of v_j in the expansion of $e^{\alpha U}\, e^{\beta D}\, v_i$

$$e^{\alpha U}\, e^{\beta D}\, v_i = \sum_j \, [i \to j]_{\alpha,\beta,z,z'} \, v_j \,.$$

A direct computation yields

$$[i \to j]_{\alpha,\beta,z,z'} = \\ \begin{cases} \dfrac{\alpha^{j-i}}{(j-i)!}\, (z+i+\frac{1}{2})_{j-i}\, F\begin{pmatrix} -z-i+\frac{1}{2}\,,\, -z'-i+\frac{1}{2} \\ j-i+1 \end{pmatrix}; \alpha\beta \Bigg)\,, & i \le j\,, \\ \dfrac{\beta^{i-j}}{(i-j)!}\, (z'+j+\frac{1}{2})_{i-j}\, F\begin{pmatrix} -z-j+\frac{1}{2}\,,\, -z'-j+\frac{1}{2} \\ i-j+1 \end{pmatrix}; \alpha\beta \Bigg)\,, & i \ge j\,, \end{cases} \tag{3.4}$$

where

$$F\left(\begin{matrix} a\,,\, b \\ c \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\, k!}\, z^k\,,$$

is the Gauss hypergeometric function.

Consider now the dual module V^* spanned by functionals v_j^* such that

$$\langle v_i^*, v_j \rangle = \delta_{ij}$$

and equipped with the dual action of $\mathfrak{sl}(2)$

$$\begin{aligned} U\, v_k^* &= -(z+k-\tfrac{1}{2})\; v_{k-1}^*\,, \\ D\, v_k^* &= -(z'+k+\tfrac{1}{2})\, v_{k+1}^*\,. \end{aligned}$$

Denote by $[i \to j]^*_{\alpha,\beta,z,z'}$ the coefficient of v_j^* in the expansion of $e^{\alpha U}\, e^{\beta D}\, v_i^*$

$$e^{\alpha U}\, e^{\beta D}\, v_i^* = \sum_j \, [i \to j]^*_{\alpha,\beta,z,z'} \, v_j^* \,.$$

We have

$$[i \to j]^*_{\alpha,\beta,z,z'} = \\ \begin{cases} \dfrac{(-\beta)^{j-i}}{(j-i)!}\, (z'+i+\frac{1}{2})_{j-i}\, F\left(\begin{matrix} z+j+\frac{1}{2}\,,\, z'+j+\frac{1}{2} \\ j-i+1 \end{matrix}; \alpha\beta \right)\,, & i \le j\,, \\ \dfrac{(-\alpha)^{i-j}}{(i-j)!}\, (z+j+\frac{1}{2})_{i-j}\, F\left(\begin{matrix} z+i+\frac{1}{2}\,,\, z'+i+\frac{1}{2} \\ i-j+1 \end{matrix}; \alpha\beta \right)\,, & i \ge j\,, \end{cases} \tag{3.5}$$

3.2 Remarks

3.2.1 Periodicity

Observe that representations whose parameters z and z' are related by the tranformation

$$(z, z') \mapsto (z + m, z' + m), \quad m \in \mathbb{Z},$$

are equivalent. The above transformation amounts to just a renumeration of the vectors v_k. G. Olshanski pointed out that this periodicity in (z, z') is reflected in a similar periodicity of various asymptotic properties of z-measures, see Sections 10 and 11 of [3].

3.2.2 Unitarity

Recall that the z-measures are positive if either $z' = \bar{z}$ or $z, z' \in (n, n+1)$ for some n. By analogy with representation theory of $SL(2)$, these cases were called the principal and the complementary series.

Observe that in these case the above representations have a positive defined Hermitian form Q which is invariant in the following sense

$$Q(Lu, v) = Q(u, Lv), \quad Q(Uu, v) = Q(u, Dv).$$

The form Q is given by

$$Q(v_k, v_k) = \begin{cases} 1 & z' = \bar{z}, \\ \dfrac{\Gamma(z' + k + \frac{1}{2})}{\Gamma(z + k + \frac{1}{2})} & z, z' \in (n, n+1), \end{cases}$$

and $Q(v_k, v_l) = 0$ if $k \neq l$. It follows that the operators

$$\tfrac{i}{2} L, \tfrac{1}{2}(U - D), \tfrac{i}{2}(U + D) \in \mathfrak{sl}(2),$$

which form a standard basis of $\mathfrak{su}(1,1)$, are skew-Hermitian and hence this representation of $\mathfrak{su}(1,1)$ can be integrated to a unitary representation of the universal covering group of $SU(1,1)$. This group $SU(1,1)$ is isomorphic to $SL(2,\mathbb{R})$ and the above representations correspond to the principal and complementary series of unitary representations of the universal covering of $SL(2,\mathbb{R})$, see [13].

3.3 The infinite wedge module

Consider the module $\Lambda^{\frac{\infty}{2}} V$ which is, by definition, spanned by vectors

$$\delta_S = v_{s_1} \wedge v_{s_2} \wedge v_{s_3} \wedge \dots ,$$

where $S = \{s_1 > s_2 > \dots\} \subset \mathbb{Z} + \frac{1}{2}$ is a such subset that both sets

$$S_+ = S \setminus \left(\mathbb{Z}_{\leq 0} - \tfrac{1}{2}\right) , \quad S_- = \left(\mathbb{Z}_{\leq 0} - \tfrac{1}{2}\right) \setminus S$$

are finite. We equip this module with the inner product in which the basis $\{\delta_S\}$ is orthonormal. Introduce the following operators

$$\psi_k, \psi_k^* : \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V .$$

The operator ψ_k is the exterior multiplication by v_k

$$\psi_k\,(f) = v_k \wedge f .$$

The operator ψ_k^* is the adjoint operator; it can be also given by the formula

$$\psi_k^*\,(v_{s_1} \wedge v_{s_2} \wedge v_{s_3}) = \sum_i (-1)^{i+1} \langle v_k^*, v_{s_i} \rangle\, v_{s_1} \wedge v_{s_2} \wedge \dots \wedge \widehat{v_{s_i}} \wedge \dots .$$

These operators satisfy the canonical anticommutation relations

$$\psi_k \psi_k^* + \psi_k^* \psi_k = 1 ,$$

all other anticommutators being equal to 0. It is clear that

$$\psi_k \psi_k^*\, \delta_S = \begin{cases} \delta_S\,, & k \in S\,, \\ 0\,, & k \notin S\,. \end{cases} \tag{3.6}$$

A general reference on the infinite wedge space is Chapter 14 of the book [8].

The Lie algebra $\mathfrak{sl}(2)$ acts on $\Lambda^{\frac{\infty}{2}} V$. The action of U and D are the obvious extensions of the action on V. In terms of the fermionic operators ψ_k and ψ_k^* they can be written as follows

$$U = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \left(z + k + \tfrac{1}{2}\right)\, \psi_{k+1} \psi_k^* ,$$

$$D = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \left(z' + k + \tfrac{1}{2}\right)\, \psi_k \psi_{k+1}^* .$$

The easiest way to define the action of L is to set it equal to $[D, U]$ by definition. We obtain

$$L = 2H + (z + z')\, C + zz' \,,$$

where H is the energy operator

$$H = \sum_{k>0} k\, \psi_k \psi_k^* - \sum_{k<0} k\, \psi_k^* \psi_k \,,$$

and C is the charge

$$C = \sum_{k>0} \psi_k \psi_k^* - \sum_{k<0} \psi_k^* \psi_k \,.$$

It is clear that

$$C\, \delta_S = (|S_+| - |S_-|)\ \delta_S$$

and, similarly,

$$H\, \delta_S = \left(\sum_{k \in S_+} k - \sum_{k \in S_-} k \right)\ \delta_S \,.$$

The charge is preserved by the $\mathfrak{sl}(2)$ action.

Consider the zero charge subspace, that is, the kernel of C

$$\Lambda_0 \subset \Lambda^{\frac{\infty}{2}} V \,.$$

It is spanned by vectors which, abusing notation, we shall denote by

$$\delta_\lambda = \delta_{S(\lambda)} \,, \quad S(\lambda) = \left\{ \lambda_1 - \tfrac{1}{2}, \lambda_2 - \tfrac{3}{2}, \lambda_3 - \tfrac{5}{2}, \dots \right\} \,, \tag{3.7}$$

where λ is a partition. One immediately sees that the action of $\mathfrak{sl}(2)$ on $\{\delta_\lambda\}$ is identical with Kerov's operators.

3.4 Correlation functions

Recall that the correlation functions were defined by

$$\rho(X) = \mathcal{M}\big(\{\lambda, X \subset \mathfrak{S}(\lambda)\}\big) \,,$$

where the finite set

$$X = \{x_1, \dots, x_s\} \subset \mathbb{Z} + \tfrac{1}{2}$$

is arbitrary.

The important observation is that (2.4) and (3.6) imply the following expression for the correlation functions

$$\rho(X) = (1-\xi)^{zz'} \left(e^{\sqrt{\xi}\, D} \prod_{x\in X} \psi_x \psi_x^* \, e^{\sqrt{\xi}\, U}\, \delta_\emptyset, \delta_\emptyset \right) . \tag{3.8}$$

We apply to (3.8) the same strategy we applied to (2.5) which is to commute the operators $e^{\sqrt{\xi}\, D}$ and $e^{\sqrt{\xi}\, U}$ all the way to the right and left, respectively, and then use (2.6). From (2.7), we have for any operator A the following identity

$$\begin{aligned} e^{\beta\, D}\, A\, e^{\alpha\, U} &= \\ & e^{\frac{\alpha}{1-\alpha\beta}\, U} \left[e^{-\frac{\alpha}{1-\alpha\beta}\, U}\, e^{\beta\, D}\, A\, e^{-\beta\, D} e^{\frac{\alpha}{1-\alpha\beta}\, U} \right] (1-\alpha\beta)^{-L} e^{\frac{\beta}{1-\alpha\beta}\, D} . \end{aligned} \tag{3.9}$$

We now apply this identity with $\alpha = \beta = \sqrt{\xi}$ and $A = \prod \psi_x \psi_x^*$ to obtain

$$\rho(X) = \left(G \prod_{x\in X} \psi_x \psi_x^* \, G^{-1}\, \delta_\emptyset, \delta_\emptyset \right) , \tag{3.10}$$

where

$$G = \exp\left(\frac{\sqrt{\xi}}{\xi - 1}\, U \right) \exp\left(\sqrt{\xi}\, D \right) .$$

Consider the following operators

$$\Psi_k = G\, \psi_k\, G^{-1} = \sum_i\, [k \to i]\ \psi_i \,, \tag{3.11}$$

$$\Psi_k^* = G\, \psi_k^*\, G^{-1} = \sum_i\, [k \to i]^*\ \psi_i^* \,, \tag{3.12}$$

with the understanding that matrix elements without parameters stand for the following choice of parameters

$$[k \to i] = [k \to i]_{\xi^{1/2}(\xi-1)^{-1},\, \xi^{1/2}, z, z'} \,, \tag{3.13}$$

and with same choice of parameters for $[k \to i]^*$. The first equality in both (3.11) and (3.12) is a definition and the second equality follows from the

definition of the operators ψ_i and the definition of the matrix coefficients $[i \to j]_{\alpha,\beta,z,z'}$.

From (3.10) we obtain

$$\rho(X) = \left(\prod_{x \in X} \Psi_x \Psi_x^* \, \delta_\emptyset, \delta_\emptyset \right) . \tag{3.14}$$

Applying Wick's theorem to (3.14), or simply unraveling the definitions in the right-hand side of (3.14), we obtain the following

Theorem 1. *We have*

$$\rho(X) = \det \left[K(x_i, x_j) \right]_{1 \le i,j \le s} , \tag{3.15}$$

where the kernel K is defined by

$$K(i,j) = \left(\Psi_i \Psi_j^* \, \delta_\emptyset, \delta_\emptyset \right) .$$

Observe that

$$(\psi_l \, \psi_m^* \, \delta_\emptyset, \delta_\emptyset) = \begin{cases} 1 \, , & l = m < 0 \, , \\ 0 \, , & \text{otherwise} \, . \end{cases}$$

Therefore, applying the formulas (3.11) and (3.12) we obtain

Theorem 2. *We have*

$$K(i,j) = \sum_{m = -1/2, -3/2, \dots} [i \to m] \; [j \to m]^* , \tag{3.16}$$

with the agreement (3.13) *about matrix elements without parameters.*

The formula (3.16) is the analog of the Proposition 2.9 in [2] for the discrete Bessel kernel.

We conclude this section with the following formula which, after substituting the formulas (3.4) and (3.5) for matrix elements, becomes the formula of Borodin and Olshanski [4].

Theorem 3. *We have*

$$K(i,j) = \frac{z' \sqrt{\xi} \left[i \to \frac{1}{2} \right] \; \left[j \to -\frac{1}{2} \right]^* - z \frac{\sqrt{\xi}}{(\xi - 1)^2} \left[i \to -\frac{1}{2} \right] \; \left[j \to \frac{1}{2} \right]^*}{i - j} , \tag{3.17}$$

where for $i = j$ the right-hand side is defined by continuity.

More generally, set

$$K(i,j)_{\alpha,\beta} = \left(\Psi_i(\alpha,\beta)\,\Psi_j^*(\alpha,\beta)\,\delta_\emptyset,\delta_\emptyset\right)\,.$$

where

$$\begin{aligned}\Psi_k &= e^{\alpha U}\,e^{\beta D}\,\psi_k\,e^{-\beta D}\,e^{-\alpha U} = \sum_i\,[k\to i]_{\alpha,\beta,z,z'}\;\psi_i\\ \Psi_k^* &= e^{\alpha U}\,e^{\beta D}\,\psi_k^*\,e^{-\beta D}\,e^{-\alpha U} = \sum_i\,[k\to i]^*_{\alpha,\beta,z,z'}\;\psi_i^*\,.\end{aligned}$$

We will prove that

$$\begin{aligned}K(i,j)_{\alpha,\beta} = \Big(\beta z'\,&\left[i\to\tfrac12\right]_{\alpha,\beta,z,z'}\;\left[j\to-\tfrac12\right]^*_{\alpha,\beta,z,z'} -\\ &\alpha(\alpha\beta-1)z\left[i\to-\tfrac12\right]_{\alpha,\beta,z,z'}\;\left[j\to\tfrac12\right]^*_{\alpha,\beta,z,z'}\Big)\,/(i-j)\,.\end{aligned}\tag{3.18}$$

First, we treat the case $i\neq j$ in which case we can clear the denominators in (3.18). From the following computation with 2×2 matrices

$$\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-\beta&1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\\beta&1\end{pmatrix}\begin{pmatrix}1&-\alpha\\0&1\end{pmatrix} = \begin{pmatrix}1-2\alpha\beta&2\alpha(\alpha\beta-1)\\-2\beta&2\alpha\beta-1\end{pmatrix}$$

we conclude that

$$e^{\alpha U}\,e^{\beta D}\,Le^{-\beta D}\,e^{-\alpha U} = L+T\,,$$

where

$$T = -2\alpha\beta\,L + 2\beta\,D + 2\alpha(\alpha\beta-1)\,U\,.$$

This can be rewritten as follows

$$[L,e^{\alpha U}\,e^{\beta D}] = -T\,e^{\alpha U}\,e^{\beta D}\,,\tag{3.19}$$

$$[L,e^{-\alpha U}\,e^{-\beta D}] = e^{-\alpha U}\,e^{-\beta D}\,T\,.\tag{3.20}$$

Note that

$$[L,\psi_i\,\psi_j^*] = 2(i-j)\,\psi_i\,\psi_j^*\,.\tag{3.21}$$

From (3.19), (3.20), and (3.21) we have

$$\begin{aligned}\big[L, \Psi_i(\alpha,\beta)\, \Psi_j^*(\alpha,\beta)\big] = \\ - \big[T, \Psi_i(\alpha,\beta)\, \Psi_j^*(\alpha,\beta)\big] + 2(i-j)\, \Psi_i(\alpha,\beta)\, \Psi_j^*(\alpha,\beta)\,. \quad (3.22)\end{aligned}$$

Since $\delta_\emptyset$ is an eigenvector of L we have

$$\Big(\big[L, \Psi_i(\alpha,\beta)\, \Psi_j^*(\alpha,\beta)\big]\ \delta_\emptyset, \delta_\emptyset\Big) = 0$$

Expand this equality using (3.22) and the relations

$$\begin{aligned} T\,\delta_\emptyset &= -2\alpha\beta z z' \delta_\emptyset + 2\alpha(\alpha\beta-1) z \delta_\Box\,, \\ T^*\delta_\emptyset &= -2\alpha\beta z z' \delta_\emptyset + 2\beta z' \delta_\Box\,, \end{aligned}$$

where T^* is the operator adjoint to T and $\delta_\Box$ is the vector corresponding to the partition $(1, 0, 0, \dots)$. We obtain

$$\begin{aligned}(i-j) K(i,j)_{\alpha,\beta} = \beta z' \ \Big(\Psi_i(\alpha,\beta)\, \Psi_j^*(\alpha,\beta)\, \delta_\emptyset, \delta_\Box\Big) - \\ \alpha(\alpha\beta-1) z \ \Big(\Psi_i(\alpha,\beta)\, \Psi_j^*(\alpha,\beta)\, \delta_\Box, \delta_\emptyset\Big)\end{aligned}$$

In order to obtain (3.18) for $i \neq j$, it now remains to observe that

$$\begin{aligned} (\psi_l\, \psi_m^*\, \delta_\emptyset, \delta_\Box) &= \begin{cases} 1\,, & l = \frac{1}{2}\,,\ m = -\frac{1}{2}\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \\ (\psi_l\, \psi_m^*\, \delta_\Box, \delta_\emptyset) &= \begin{cases} 1\,, & l = -\frac{1}{2}\,,\ m = \frac{1}{2}\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \end{aligned}$$

In the case $i = j$ we argue by continuity. It is clear from (3.16) that $K(i,j)$ is an analytic function of i and j and so is the right-hand side of (3.17). The passage from (3.16) to (3.17) is based on the fact that the product i times $[i \to m]_{\alpha,\beta,z,z'}$ is a linear combination of $[i \to m]_{\alpha,\beta,z,z'}$ and $[i \to m \pm 1]_{\alpha,\beta,z,z'}$ with coefficients which are linear functions of m. Since the matrix coefficients are, essentially, the hypergeometric function, such a relation must hold for any i, not just half-integers. Hence, (3.16) and (3.17) are equal for any $i \neq j$, not necessarily half-integers. Therefore, they are equal for $i = j$.

3.5 Rim-hook analogs

The same principles apply to rim-hook analogs of the z-measures which were also considered by S. Kerov [9].

Recall that a rim hook of a diagram λ is, by definition, a skew diagram λ/μ which is connected and lies on the rim of λ. Here connected means that the squares have to be connected by common edges, not just common vertices. Rim hooks of a diagram λ are in the following 1-1 correspondence with the squares of λ: given a square $\square \in \lambda$, the corresponding rim hook consists of all squares on the rim of λ which are (weakly) to the right of and below $\square$. The length of this rim hook is equal to the hook-length of $\square$.

The entire discussion of the previous section applies to the more general operators

$$\begin{aligned} U_r\, v_k &= \left(z + \tfrac{k}{r} + \tfrac{1}{2}\right) v_{k+r}\,, \\ L_r\, v_k &= \left(\tfrac{2k}{r} + z + z'\right) v_k\,, \\ D_r\, v_k &= \left(z' + \tfrac{k}{r} - \tfrac{1}{2}\right) v_{k-r}\,, \end{aligned}$$

which satisfy the same $\mathfrak{sl}(2)$ commutation relations. The easiest way to check the commutation relations is to consider $\frac{k}{r}$ rather than k as the index of v_k; the above formulas then become precisely the formulas (3.1). The operator U_r acts on the basis $\{\delta_\lambda\}$ as follows

$$U_r\, \delta_\lambda = \sum_{\mu=\lambda+\text{ rim hook}} (-1)^{\text{height}+1} \left(z + \frac{1}{r^2} \sum_{\square\in\text{rim hook}} c(\square) \right) \delta_\mu\,, \tag{3.23}$$

where the summation is over all partitions μ which can be obtained from λ by adding a rim hook of length r, height is the number of horizontal rows occupied by this rim hook and $c(\square)$ stands, as usual for the content of the square $\square$. Similarly, the operator D_r removes rim hooks of length r. These operators were considered by Kerov [9].

It is clear that the action of the operators $e^{\alpha U_r}$ and $e^{\beta D_r}$ on a half-infinite wedge product like

$$v_{s_1} \wedge v_{s_2} \wedge v_{s_3} \wedge \dots\,,$$

essentially (up to a sign which disappears in formulas like (3.8)) factors into the tensor product of r separate actions on

$$\bigwedge_{s_i \equiv k+\frac{1}{2} \mod r} v_{s_i}\,, \quad k = 0, \dots, r-1\,.$$

Consequently, the analogs of the correlation functions (3.8) have again a determinantal form with a certain kernel $K_r(i,j)$ which has the following structure. If $i \equiv j \mod r$ then $K_r(i,j)$ is essentially the kernel $K(i,j)$ with rescaled arguments. Otherwise, $K_r(i,j) = 0$.

This factorization of the action on $\Lambda^{\frac{\infty}{2}}V$ is just one more way to understand the following well-known phenomenon. Let $\mathbb{Y}_r$ be the partial ordered set formed by partitions with respect to the following ordering: $\mu \le_r \lambda$ if μ can be obtained from λ by removing a number of rim hooks with r squares. The minimal elements of $\mathbb{Y}_r$ are called the r-cores. The r-cores are precisely those partitions which do not have any hooks of length r. We have

$$\mathbb{Y}_r \cong \bigsqcup_{r\text{-cores}} (\mathbb{Y}_1)^r \tag{3.24}$$

as partially ordered sets. Here the Cartesian product $(\mathbb{Y}_1)^r$ is ordered as follows:

$$(\mu_1, \dots, \mu_r) \le (\lambda_1, \dots, \lambda_r) \quad \Leftrightarrow \quad \mu_i \le_1 \lambda_i, \quad i = 1, \dots, r\,,$$

and the partitions corresponding to different r-cores are incomparable in the $\le_r$-order. Combinatorial algorithms which materialize the isomorphism (3.24) are discussed in Section 2.7 of the book [7]. The r-core and the r-tuple of partitions which the isomorphism (3.24) associates to a partition λ are called the r-core of λ and the r-quotient of λ. Among more recent papers dealing with r-quotients let us mention [6] where an approach similar to the use of $\Lambda^{\frac{\infty}{2}}V$ is employed, an analog of the Robinson-Schensted algorithm for $\mathbb{Y}_r$ is discussed, and further references are given.

The factorization (3.24) and the corresponding analog of the Robinson-Schensted algorithm play the central role in the recent paper [1], see also [14].

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